FINITE SPACES AND FINITE MODELS TALK

DORON GROSSMAN-NAPLES

Abstract. When we try to model simplicial complexes using posets, finite spaces arise as a natural bridge between these two categories. In this talk, I will describe the theory of these spaces and the nature of this correspondence, and discuss the resulting theory of finite models.

Contents

1. Introduction

The basic references for this talk are [9] and [2]. I also draw some from [7], which is an elementary introduction to the subject of finite spaces. A more advanced and comprehensive introduction can be found in [1].

We would like to model simplicial complexes using posets. To do this, we introduce an intermediate object: a certain type of space with a nice duality property.

Definition 1.1. An *Alexandroff space* or *A-space* is a space whose topology is closed under arbitrary intersections.

We note in particular that finite spaces are Alexandroff. There are a few pieces of notation associated with A-spaces.

Definition 1.2. The *open hull* of a set M in an Alexandroff space is the smallest open set containing M, denoted $U(M)$. We write U_x for the open hull of a singleton, and $\hat{U}_x$ for $U_x \setminus \{x\}$.

This allows us to begin making the connection with posets. Given an A-space X, we define a preorder on X by saying $x \leq y$ if $x \in U_y$. It is easily checked that this is reflexive and transitive. Conversely, given any preordered set X, we can define a topology on X by taking the open sets to be those which are downward-closed.

Theorem 1.3. *This defines an equivalence of categories from A-spaces to preorders.*

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Proof. It is enough to check that a map is continuous iff it is order-preserving. Let X, Y be preordered sets with the induced A-space structure. Then Y has basis $\{U_y\}$. Take $f : X \to Y$. The preimage of U_y is precisely those elements x such that $f(x) \leq y$. If f is monotonic, then this set is downward closed. Conversely, take $x_0, x_1 \in X$ with $x_0 \leq x_1$. If f is continuous, it follows that x_0 is in the preimage of $U_{f(x_1)}$, so $f(x_0) \leq f(x_1)$. □

Recall that a space is T_0 if given two points x and y, one has an open neighborhood not containing the other. This is precisely the same as antisymmetry in the context of A-spaces.

Corollary 1.4. *This defines an equivalence from T_0 A-spaces to posets. Moreover, it defines an equivalence from finite spaces to finite preorders, and T_0 finite spaces to finite posets.*

Henceforth, I use these terms interchangably. Observe that we can now equivalently define U_x as the set of points less than or equal to x, and $\hat{U}_x$ as the set of points strictly less than x. We dually define F_x as the set of points greater than or equal to x, and $\hat{F}_x = F_x \setminus \{x\}$. It is a nice exercise to check that F_x is the closure of $\{x\}$, and more generally that the opposite topology on X is the same as the opposite order on X.

Remark 1.5. It is easy to see that a T_1 A-space is discrete.

Remark 1.6. I take a moment to note here that the subspace topology on a subspace of an A-space is the same as the A-space topology on the associated subposet.

Since we are, after all, homotopy theorists, we are interested in the homotopy relations on maps of A-spaces. Unfortunately, the general case has some subtleties, since $\mathrm{Top}(X, Y)$ need not be Alexandroff even if both X and Y are. In the case of finite spaces, however, the mapping space is finite as well, and we can prove the following.

Theorem 1.7. *The pointwise order on* $\mathrm{Top}(X, Y)$ *coincides with the compact-open topology.*

Proof. The following proof is from [7].

Let g be a map $X \to Y$, let U_g be the open hull of g, and let $Z_g = \{f \mid f \leq g\}$. We must show these are equal. If $f \in U_g$, then for each $x \in X$, $f(x) \in U_{g(x)}$, so $f \leq g$. Conversely, if $f \leq g$, let $C \subset X$ be compact and $U \subset Y$ be open with $f(C) \subset U$; then for each $x \in C$, $g(x) \leq f(x) \in U$, so $g(x) \in U$. Thus $g(C) \subset U$. □

This tells us that two maps f, g of finite spaces are homotopic iff there is a chain of maps $f = f_0, f_1, \ldots, f_n = g$ with $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$ for each i, since a path in an A-space is a sequence of comparable points. It is true for arbitrary A-spaces that such a chain of maps induces a homotopy, but not the converse. This is because the Alexandroff topology is in general finer than the compact-open topology ([6]).

It is convenient for obvious reasons to work with T_0 A-spaces specifically. Fortunately, we may do so due to the following theorem.

Theorem 1.8. *Let X be an A-space, and define a relation $\sim$ by $x \sim y$ if $x \leq y$ and $y \leq x$. Then $X/\sim$ is an A-space with the quotient order which is T_0, and the quotient map is a homotopy equivalence.*

Proof. The following proof is adapted from [9].

Let f denote the quotient map. First, one observes that $f(U_x) = U_{f(x)}$ and $f^{-1}(U_{f(x)}) = U_x$; thus $x \leq y$ iff $f(x) \leq f(y)$. Now take $g : X/\sim \to X$ which sends x to any point in $f^{-1}(x)$. Then g is order-preserving, hence continuous; fg is the identity; and gf is less than or equal to the identity. $\square$

It is easy to check that this gives a full functor from preorders to posets. Henceforth I assume all A-spaces are T_0.

2. Relations with homotopy theory

The basic reference for this section is [9].

So far, I've talked about posets, so now it's time to talk about the relation to ordinary spaces–namely, simplicial complexes and CW complexes. To relate these to A-spaces, we'll need a lemma. An open cover is called *basis-like* if it forms a basis for a topology weaker than the given one; that is, any finite intersection of sets in the cover is a union of sets in the cover.

Lemma 2.1 (Locality of weak equivalence)**.** *Let $f : X \to Y$ be a map, and let $\mathcal{O}$ be a basis-like open cover of Y. Suppose that for each $U \in \mathcal{O}$, $f : f^{-1}(U) \to U$ is a weak equivalence. Then f is a weak equivalence.*

The hard part is proving this lemma for a cover by two sets and their intersection, which is a theorem in §10.7 of [8]. Using this lemma, we can turn A-spaces into simplicial complexes and vice versa.

Definition 2.2. Let X be a poset. The *order complex* of X, $\mathcal{K}(X)$, is the simplicial complex whose simplices are chains in X. $\mathcal{K}$ acts on monotonic maps in the obvious way.

Definition 2.3. Let K be a simplicial complex. The *face poset* of X, $\mathcal{X}(K)$, is the poset of simplices of K under inclusion. $\mathcal{X}$ acts on simplicial maps in the obvious way.

We will define natural maps between these spaces which will turn out to be weak homotopy equivalences. (In particular, this will imply that every finite poset is weak homotopy equivalent to its opposite.) Since every space is weak homotopy equivalent to a simplicial complex, which is often finite in cases of interest, this suggests that our theory is actually quite broad in its scope. (In fact, our results about face posets can be extended to regular CW complexes, but I'll restrict to simplicial complexes here for simplicity.)

Firstly, let X be a poset. We have a map $f : |\mathcal{K}(X)| \to X$ given by sending each open simplex to its smallest vertex; that is, a point in the interior of $(x_1, \ldots, x_n)$ is sent to $\min_i x_i$. Then f is continuous, since the preimage of a downwards-closed set is the union of all the open simplices with a vertex in that set. Note also that f is a natural transformation. Observe that $\{U_x \mid x \in X\}$ is a basis-like open cover, and each U_x deformation retracts to x. We can now apply Lemma 2.1, noting that $f^{-1}(U_x)$ deformation retracts to the vertex associated to x, to conclude that f is a weak homotopy equivalence.

Second, let K be a simplicial complex. To construct our natural weak equivalence $g : |K| \to \mathcal{X}(K)$, we observe that $\mathcal{K}\mathcal{X}$ is in fact the barycentric subdivision functor. (This is something you just have to think about for a little bit to believe.)

Thus applying the previous construction and composing with the subdivision homeomorphism gives us our g.

Now that we've established a connection between A-spaces and simplicial complexes, it would be nice to look at a few examples. For this, we use a tool called the non-Hausdorff suspension. This is, as one might expect, an endofunctor $\mathcal{S}$ of (finite) (T_0) A-spaces which is naturally weak homotopy equivalent to ordinary unreduced suspension. The construction is simple enough: one adjoins two additional points a and b (which you can think of as the vertices of a double cone) to X, defining both to be greater than all the points in X and not comparable to each other. This non-Hausdorff suspension is the union of two non-Hausdorff cones $\mathcal{C}X$, which are the same construction with one point rather than two. (Note that since the non-Hausdorff cone has a maximal point, it is contractible as we would expect.) Now we define a natural map $\gamma_X : SX \to \mathcal{S}X$ by sending (x, t) to x if $-1 < t < 1$, to a if $t = -1$, and to b if $t = 1$. That this map is continuous and a weak homotopy equivalence can be proven by methods similar to those used above. The details can be found in §3.4 of [7].

Now we can present the examples. Firstly, the n^{th} finite sphere is simply the n^{th} non-Hausdorff suspension of S^0. It looks like a tower of pairs of points having height $n + 1$. Second, one can apply the face poset construction to regular CW structures on various closed surfaces to obtain finite models of these which have height 3; see [5] for a general construction.

3. Minimal finite models

In the previous sections, we gave constructions for transforming between ordinary spaces and finite spaces. Now we will formalize this notion.

Definition 3.1. Let X be a space. A *finite model* of X is a finite T_0 space which is weak homotopy equivalent to X.

Remark 3.2. I take a moment to note here that weak homotopy equivalence is the best we can hope for here: by Theorem 1.3 of [2], a connected T_1 space homotopy equivalent to a finite space is contractible.

Our previous constructions therefore give us a way of producing a finite model of any regular CW complex with finitely many cells. Of course, there will in general be many finite models for a given space; for example, simply take repeated barycentric subdivisions of a finite regular CW complex, then take the face poset. It is interesting to try to find which of these finite models is the smallest. Unsurprisingly, this is not easy. It is, after all, a problem of classification of all finite posets up to weak equivalence. It turns out, however, that the related problem of reducing a finite space to a space minimal in its homotopy class is quite simple. The theory in this section was originally developed in [10] and uses the terminology made standard by May in [7].

Definition 3.3. A *beat point* for a finite poset X is a point $x \in X$ such that $\hat{U_x}$ has a maximum or $\hat{F_x}$ has a minimum.

If x is a beat point, say with y the maximum of its underset or minimum of its overset, then the map $X \to X \setminus \{x\}$ sending x to y and fixing all other points is a deformation retract. (It is either greater or less than the identity on X.) Therefore, we can remove a beat point of a finite space without changing its homotopy type.

When we do this repeatedly until no beat points remain, the resulting space is called the *core* of X. I will call a finite space with no beat points *Stong minimal*. (The standard practice is to call such spaces simply *minimal*, but this results in a conflict of terminology, as we will see shortly.)

Proposition 3.4. *Let X be a Stong minimal finite space and $f : X \to X$ a map homotopic to the identity. Then f is the identity.*

Proof. It is enough to show that if $f \geq \mathrm{id}$ or $f \leq \mathrm{id}$ then $f = \mathrm{id}$. We consider the first case. Certainly f fixes all maximal points. Now inductively assume f fixes each element of $\hat{U}_x$ for some x. Then $f(x) \geq x$, but $f(x)$ is also a lower bound for $\hat{U}_x$. Since X has no beat points, it follows that $f(x) = x$, so induction shows f is the identity. The case $f \leq \mathrm{id}$ is similar. □

Corollary 3.5. *Two finite spaces are homotopy equivalent iff their cores are homeomorphic.*

Proof. Certainly spaces with homeomorphic cores are homotopy equivalent. Conversely, a homotopy equivalence between finite spaces restricts to a homotopy equivalence between their cores, which is a homeomorphism by the previous proposition. □

We see therefore that a homotopy equivalence between finite spaces consists of removing and adding beat points. Under the correspondence with simplicial complexes (and regular CW complexes), this corresponds to so-called simple homotopy equivalence. A simple homotopy equivalence is a homotopy equivalence of CW complexes obtained by collapsing and expanding cells. Simple homotopy equivalences have an interesting theory of their own, including things like an obstruction to simplicity for ordinary homotopy equivalences and a variant of the s-cobordism theorem. This theory is beyond the scope of this talk, but it is developed in [12] and exposited in [4].

I have given a description of some Stong minimal finite models for all closed surfaces in [5] which are obtained as the face posets of regular CW structures. Specifically, we have the following.

Theorem 3.6. *The orientable closed surface of genus g has a Stong minimal finite model with $14g+2$ points. The non-orientable closed surface of genus g has a Stong minimal finite model with $11g+2$ points.*

We have described a general algorithm for classifying finite spaces up to weak equivalence, as well as for reducing them to the smallest poset in their homotopy class. The corresponding problem for weak homotopy is more subtle, and no general algorithm is known. However, some techniques have been developed for dealing with specific cases.

Definition 3.7. A finite space is *absolutely minimal* if it is minimal in its weak homotopy class.

Remark 3.8. The standard term for such an object is “minimal finite model”, but this and the standard term for a Stong minimal finite space lead to such oddities as spaces which are finite models and minimal finite spaces but not minimal finite models.

The first result comes from [2].

Theorem 3.9. *For each n, $\mathcal{S}^n S_0$ is the unique absolutely minimal finite model of S^n.*

There are a few more finite models found (constructively) using the technique of "poset splitting" in [3].

Theorem 3.10. *There are exactly two absolutely minimal finite models of $\mathbb{T}^2$ with 16 points, one of which is $\mathcal{S}S^1 \times \mathcal{S}S^1$, and both are self-opposite. There are exactly two absolutely minimal finite models of $\mathbb{RP}^2$ with 13 points, which are opposite to each other. There are exactly two absolutely minimal finite models of $\mathbb{K}^2$ with 16 points, which are opposite to each other.*

Beyond these specific examples, I am not aware of any nontrivial examples known of absolutely minimal finite models. There are some techniques for reduction of finite spaces up to weak equivalence ([11]), but these are known not to be sufficient for constructing absolutely minimal finite models. That said, some lower bounds for the sizes of these models, or for particular kinds of models, have been derived. The following results are from [5]. I denote the cardinality of a set S by $\#S$ to avoid confusion with geometric realization.

Theorem 3.11. *Let X be a finite model of a closed surface S other than S^2 or $\mathbb{RP}^2$. Then $\#X \geq \max(16, \log_2(|\chi(S)|))$. If X has height 3, then $\#X \geq \sqrt{2|\chi(S) - 7|}$.*

The interest of finite models having height 3 is that this includes all models arising from regular CW structures on closed surfaces, as well as all of those whose order complexes are surfaces. While it is not completely obvious, these two conditions are equivalent.

Definition 3.12. A *finite manifold* is a finite space whose order complex is a topological manifold. A *finite surface* is a finite space whose order complex is a topological surface. A *minimal finite surface* is a finite space which is minimal among finite surfaces in its weak homotopy class.

Theorem 3.13. *A finite space is a finite surface iff it is the face poset of a regular CW structure on a closed surface.*

If we assume this condition on our posets, we can get some strong lower bounds.

Theorem 3.14. *Let X be a finite surface modelling a closed surface S of genus g. If S is orientable, $\#X \geq 2\lceil 4\sqrt{g}\rceil + 2g + 6$. If S is non-orientable, $\#X \geq 2\lceil 2\sqrt{2g}\rceil + g + 6$.*

It is possible to construct a finite orientable surface of genus g achieving this bound in the case that g is a perfect square, or more generally when it is the product of two integers that are sufficiently close.

4. Bibliography

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